

COLIMITS OF ABELIAN GROUPS

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ABSTRACT. In this paper we study the colimit $N_2(G)$ of abelian subgroups of a discrete group G . This group is the fundamental group of a subspace $B(2, G)$ of the classifying space BG as introduced in [1]. We describe $N_2(G)$ for certain groups, and apply our results to study the homotopy type of the space $B(2, G)$. We give a list of classes of groups for which $B(2, G)$ is not an Eilenberg–MacLane space of type $K(\pi, 1)$.

1. INTRODUCTION

It is well known that the classifying space BG of a discrete group G can be described as a simplicial set where the n -simplices are ordered n -tuples of elements in the group. It is possible to introduce certain simplicial subsets by putting conditions on the ordered n -tuples which respect the simplicial structure. An example of such a construction as given in [1] is to consider the simplicial set $B(q, G)$ whose n -simplices consist of n -tuples (g_1, g_2, \dots, g_n) of elements in G such that these elements generate a subgroup $\langle g_1, g_2, \dots, g_n \rangle \subset G$ of nilpotency class less than q . For example $B(2, G)$ consists of n -tuples of pairwise commuting elements. Equivalently, we can say that the set of n -simplices is the space of homomorphisms $\text{Hom}(\mathbb{Z}^n, G) \subset G^n$. This definition also works more generally for topological groups where the space of homomorphisms inherits a topology from the direct product G^n . For a recent survey on spaces of homomorphisms and related constructions such as $B(q, G)$ see [4]. A key property of BG is that it classifies principal G -bundles; there is a bijection between the set of isomorphism classes of principal G -bundles over a CW-complex X and the set of homotopy classes of maps $X \rightarrow BG$. It is shown in [2] that $B(q, G)$ is a classifying space for principal G -bundles of certain type, so called principal G -bundles of transitional nilpotency class less than q . It is a natural problem to study the homotopy type of $B(q, G)$. It is not always true that for a discrete group G the space $B(q, G)$ is an Eilenberg–MacLane space of type $K(\pi, 1)$, as opposed to the case of the usual classifying space. It was conjectured in [1, page 15] that for finite groups $B(q, G)$ is a $K(\pi, 1)$ space, a natural expectation as a consequence of the types of groups studied there. Extraspecial 2-groups appeared as the first counter examples in [5]. Our approach is purely algebraic, except in the last section we apply our results to $B(q, G)$ to study its homotopical properties.

The connection to algebra is via the fundamental group. In [1] it is shown that the fundamental group of $B(q, G)$ is isomorphic to the colimit $N_q(G)$ of the nilpotent subgroups $N \subset G$ of class less than q . It turns out that some of the basic homotopy theoretic properties of $B(q, G)$ are determined by the group $N_q(G)$. Throughout the paper we focus on the case $q = 2$. Then $N_2(G)$ has the following presentation

$$\langle (g), g \in G \mid (g)(h) = (gh) \text{ if } [g, h] = 1 \rangle$$

where $[g, h] = ghg^{-1}h^{-1}$. The natural map $\epsilon : N_2(G) \rightarrow G$ induced by the inclusion $B(2, G) \subset BG$ can be described by the assignment $(g) \mapsto g$. Let $D_2(G)$ denote the kernel of the homomorphism $G \times G \rightarrow G/[G, G]$ defined by the multiplication map $(x, y) \mapsto xy[G, G]$. There is also a natural map $\bar{\epsilon} : N_2(G) \rightarrow D_2(G)$ defined by $(g) \mapsto (g, g^{-1})$, which factors ϵ . The main result of the paper essentially gives a group theoretic condition which implies that the map $\bar{\epsilon}$ is an isomorphism. This condition is given by a set of non-identity elements $\{g_i\}_{i=1}^{2r}$, which we call a symplectic sequence, satisfying the commutation rules:

$$\begin{aligned} [g_i, g_{i+r}] &= [g_j, g_{j+r}] \text{ for all } 1 \leq i, j \leq r, \\ [g_i, g_j] &= 1 \text{ for any other pair.} \end{aligned}$$

The sequence is called non-trivial if $[g_i, g_{i+r}] \neq 1$.

Theorem 1.1. *Let $\{g_i\}_{i=1}^{2r}$ be a non-trivial symplectic sequence in G for some $r \geq 2$, and S denote the subgroup generated by $\{g_i\}$. Then the natural map $N_2(S) \rightarrow D_2(S)$ is an isomorphism. Moreover $N_2(S) \rightarrow N_2(G)$ is injective, and its image is the subgroup generated by $\{(g_i)\}$.*

When G is finite this theorem gives an explicit group theoretic condition which implies that the space $B(2, G)$ is not an Eilenberg–MacLane space of type $K(\pi, 1)$.

Theorem 1.2. *Suppose that G is a finite group which has a non-trivial symplectic sequence $\{g_i\}_{i=1}^{2r}$ for some $r \geq 2$. Then $B(2, G)$ is not a $K(\pi, 1)$ space.*

Using this theorem we extend the list of groups G for which $B(2, G)$ is not a $K(\pi, 1)$ space to the following classes of groups: Extraspecial p -groups of rank ≥ 4 , general linear groups $GL_n(\mathbb{F}_q)$, $n \geq 4$, over a field of characteristic p , and symmetric groups Σ_k on k letters where $k \geq 2^4$.

The organization of the paper is as follows. In Section 2 we describe the basic properties of the groups $N_q(G)$. In the rest of the paper we mostly restrict to the case $q = 2$. Symplectic sequences are defined, and Theorem 1.1 is proved in Section 3. Applications of the algebraic results, in particular Theorem 1.2, and examples are given in Section 4.

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2. PRELIMINARIES

The descending central series of a group G is the normal series

$$1 \subset \cdots \subset \Gamma^{q+1}(G) \subset \Gamma^q(G) \subset \cdots \subset \Gamma^2(G) \subset \Gamma^1(G) = G$$

defined inductively $\Gamma^1(G) = G$, $\Gamma^{q+1}(G) = [\Gamma^q(G), G]$. A group is nilpotent of class less than q if $\Gamma^q(G) = 1$. The collection $\mathcal{N}(q, G)$ of subgroups of class less than q is a partially ordered set under inclusion. The colimit of the groups in this poset in the category of groups is denoted by

$$N_q(G) = \operatorname{colim}_{\mathcal{N}(q, G)} N.$$

The colimit comes with a collection of homomorphisms $\eta_N : N \rightarrow N_q(G)$ such that the diagram

$$\begin{array}{ccc} N_q(G) & \xrightarrow{\epsilon} & G \\ \eta_N \uparrow & \nearrow & \\ N & & \end{array}$$

commutes. In particular this implies that η_N is an inclusion for every N . There is a natural map $\epsilon : N_q(G) \rightarrow G$ induced by the inclusions $N \rightarrow G$. This map is surjective since all the cyclic subgroups $\langle g \rangle$ are contained in $\mathcal{N}(q, G)$, where $g \in G$. We denote the image of g under $\eta_{\langle g \rangle}$ simply by $\eta(g)$ or (g) . The construction $G \mapsto N_q(G)$ is natural and defines an endofunctor N_q of the category of groups **Grp**.

The group $N_q(G)$ can be constructed as a quotient of the free group $F(G)$ generated on the set $\{(g) \mid g \in G\}$. Let R_q denote the normal closure of the subgroup of $F(G)$ generated by the products $(gh)^{-1}(g)(h)$ such that the subgroup $\langle g, h \rangle \subset G$ is of class less than q . The natural map $(g) \mapsto \eta(g)$ establishes an isomorphism between $F(G)/R_q$ and $N_q(G)$. Its inverse is induced by the maps $N \rightarrow F(G)/R_q$. Therefore we obtain a presentation of the colimit

$$N_q(G) = \langle (g), g \in G \mid (g)(h) = (gh) \text{ if } \Gamma^q(\langle g, h \rangle) = 1 \rangle. \quad (1)$$

When $q = 2$ the relations remember the multiplication between the commuting elements, as q gets larger more relations are added which correspond to higher nilpotence information. In particular we have a sequence of surjections

$$N_2(G) \twoheadrightarrow N_3(G) \twoheadrightarrow \cdots \twoheadrightarrow N_q(G) \twoheadrightarrow N_{q+1}(G) \twoheadrightarrow \cdots \twoheadrightarrow G.$$

Alternatively these groups can be described by their universal property: There is a natural bijection

$$\mathbf{Grp}(\operatorname{colim}_{\mathcal{N}(q, G)} N, H) \cong \lim_{\mathcal{N}(q, G)} \mathbf{Grp}(N, H).$$

This means that a homomorphism $N_q(G) \rightarrow H$ is a set map $\phi : G \rightarrow H$ which restricts to a group homomorphism on the members of $\mathcal{N}(q, G)$, we call such a map a *nil_q-map*. Equivalently it is a set map satisfying $f(g_0)f(g_1) = f(g_0g_1)$ whenever $\Gamma^q(\langle g_0, g_1 \rangle) = 1$. Therefore enlarging the morphisms of the category of groups by allowing such maps is equivalent to studying the image of the functor N_q . Define a category **Grp_q** whose objects are groups and morphisms are nil_q-maps. Note that N_q extends to this category and is the left adjoint of the inclusion functor

$$N_q : \mathbf{Grp}_q \rightleftarrows \mathbf{Grp} : \iota_q$$

with the unit $\eta : G \rightarrow N_q(G)$ defined by $\eta(g) = \eta_{\langle g \rangle}(g)$ and the counit $\epsilon : N_q(G) \rightarrow G$. Any nil_q-map $\phi : G \rightarrow H$ induces a group homomorphism $\epsilon N_q(\phi) : N_q(G) \rightarrow H$. Note also that any nil_r-map is a nil_q-map for $r \geq q$. There is a sequence of inclusions of categories

$$\mathbf{Grp} \subset \cdots \subset \mathbf{Grp}_{q+1} \subset \mathbf{Grp}_q \subset \cdots \subset \mathbf{Grp}_3 \subset \mathbf{Grp}_2.$$

Note that if a group G is nilpotent of class less than q then every nil_q-map $G \rightarrow H$ is by definition a group homomorphism. Conversely, it is tempting to characterize groups G for which every nil_q-map $G \rightarrow H$ is a group homomorphism. We state the following conjecture as a new characterization of nilpotent groups:

Conjecture 2.1. *A group G is nilpotent of class less than q if and only if the natural map $\epsilon : N_q(G) \rightarrow G$ is an isomorphism.*

We prove the case $q = 2$ which is special since there is a distinguished class of nil_2 -maps $\omega_n : N_2(G) \rightarrow N_2(G)$ for each integer $n \in \mathbb{Z}$ defined by raising a generator to the n -th power $(g) \mapsto (g^n)$. The map ω_{-1} is an automorphism of $N_2(G)$ induced by the inversion map $i : G \rightarrow G$ which sends an element to its inverse.

Proposition 2.2. *A group G is abelian if and only if the natural map $\epsilon : N_2(G) \rightarrow G$ is an isomorphism.*

Proof. The inversion map i induces a group homomorphism ω_{-1} on $N_2(G)$ hence by the commutative diagram of nil_2 -maps

$$\begin{array}{ccc} G & \xrightarrow{i} & G \\ \downarrow \eta & & \uparrow \epsilon \\ N_2(G) & \xrightarrow{\omega_{-1}} & N_2(G) \end{array}$$

it induces a group homomorphism on G . □

To study the $q = 2$ case in more detail we introduce the group $D_2(G)$. The group $D_2(G)$ is defined to be the kernel of the multiplication map

$$G \times G \rightarrow G/[G, G], \quad (x, y) \mapsto xy[G, G].$$

Observe that the natural map $\epsilon : N_2(G) \rightarrow G$ which sends a generator $\eta(g)$ to g factors as

$$\begin{array}{ccc} N_2(G) & & \\ \downarrow \bar{\epsilon} & \searrow \epsilon & \\ D_2(G) & \xrightarrow{\pi_1} & G \end{array}$$

where $\bar{\epsilon}$ sends $\eta(g)$ to (g, g^{-1}) , and π_1 is the projection onto the first coordinate.

Proposition 2.3. *The map $\bar{\epsilon}$ is surjective, and ϵ is the composition $\pi_1 \bar{\epsilon}$.*

Proof. The kernel of the projection map $\pi_1 : D_2(G) \rightarrow G$ is the subgroup $1 \times [G, G]$. The surjectivity of $\bar{\epsilon}$ follows from the equation

$$((gh)^{-1}, gh)(g, g^{-1})(h, h^{-1}) = (1, [g, h]),$$

which implies that $D_2(G)$ is generated by the pairs of the form (g, g^{-1}) . Second part of the statement is clear from the definition of the maps. □

3. SYMPLECTIC SEQUENCES

In this section we study the group $N_2(G)$ by using the presentation 1 given in §2:

$$\langle (g) \mid (g)(h) = (gh) \text{ if } [g, h] = 1 \rangle.$$

The main result of this section is a group theoretic condition which implies that the map $\bar{e} : N_2(G) \rightarrow D_2(G)$ is an isomorphism. Next definition is essential in our study.

Definition 3.1. A sequence of elements $\{g_i\}_{i=1}^{2r}$ in G is called a *symplectic sequence* if the following conditions are satisfied

- (1) $c = [g_i, g_{i+r}] = [g_j, g_{j+r}]$ for all $1 \leq i, j \leq r$,
- (2) $[g_i, g_j] = 1$ for all $1 \leq i, j \leq 2r$ and $|i - j| \neq r$.

The sequence is called *non-trivial* if $c \neq 1$, and *trivial* if $c = 1$.

Note that the element $c = [g_i, g_{i+r}]$ commutes with all the other g_j for all j . The important fact about symplectic sequences is that they are preserved under nil_2 -maps. This will be a special case of the following computation.

Lemma 3.2. Let $\{g_i\}_{i=1}^{2r}$ be a symplectic sequence in G , and $\phi : G \rightarrow H$ a nil_2 -map. If $r \geq 2$ then for any positive integer a, b, c, d the equation

$$[\phi(g_i^a g_{i+r}^b), \phi(g_i^c g_{i+r}^d)] = [\phi(g_j), \phi(g_j)]^{ad-bc}, \quad 1 \leq i, j \leq r,$$

holds in H .

Proof. For simplicity of notation set $t_i = g_i^a g_{i+r}^b$, $t_{i+r} = g_i^c g_{i+r}^d$, and $n = ad - bc$. We will use the following observation

$$\begin{aligned} [g_j^{-n} t_{i+r}, t_i^{-1} g_{j+r}] &= [g_j^{-n}, g_{j+r}] [t_{i+r}, t_i^{-1}] \\ &= [g_j^n, g_{j+r}]^{-1} [t_i, t_{i+r}] = 1. \end{aligned}$$

Note that the commutators above lie in the subgroup $\langle c \rangle \subset G$ where c commutes with all g_i . For all $j \neq i$ the following holds

$$\begin{aligned} [\phi(t_i), \phi(t_{i+r})] &= \phi(t_i) \phi(g_j^n g_j^{-n}) \phi(t_{i+r}) \phi(t_i)^{-1} \phi(g_{j+r} g_{j+r}^{-1}) \phi(t_{i+r})^{-1} \\ &= \phi(g_j^n) \phi(t_i) \phi(g_j^{-n} t_{i+r}) \phi(t_i^{-1} g_{j+r}) \phi(t_{i+r})^{-1} \phi(g_{j+r}^{-1}) \\ &= \phi(g_j^n) \phi(t_i) \phi(t_i^{-1} g_{j+r}) \phi(g_j^{-n} t_{i+r}) \phi(t_{i+r})^{-1} \phi(g_{j+r}^{-1}) \\ &= \phi(g_j^n) \phi(g_{j+r}) \phi(g_j^n)^{-1} \phi(g_{j+r})^{-1} \\ &= [\phi(g_j^n), \phi(g_{j+r})] \\ &= [\phi(g_j), \phi(g_{j+r})]^n \end{aligned}$$

where we also make use of the identity $\phi(g)^{-1} = \phi(g^{-1})$. □

Taking $(a, b) = (1, 0)$ and $(c, d) = (0, 1)$ we obtain the following.

Corollary 3.3. Suppose that $\{g_i\}_{i=1}^{2r}$, $r \geq 2$, is a symplectic sequence in G , and $\phi : G \rightarrow H$ is a nil_2 -map. Then $\{\phi(g_i)\}_{i=1}^{2r}$ is a symplectic sequence in H .

For the rest of this section assume that G has a non-trivial symplectic sequence $\{g_i\}_{i=1}^{2r}$ where $r \geq 2$. Let $S \subset G$ denote the subgroup generated by the elements $\{g_i\}_{i=1}^{2r}$. We make some observations about the structure of S . The commutator of S is generated by $c = [g_i, g_{i+r}]$, and is contained in the center $Z(S)$. This implies that the commutator is bilinear on S : $[xy, z] = [x, z][y, z]$. Recall that we write (x) to denote the image of x under the canonical nil_2 -map $\eta : S \rightarrow N_2(S)$. According to the presentation 1 we are allowed to write

$$(x)(y) = (xy) \text{ if } [x, y] = 1.$$

In particular $(x^m) = (x)^m$ for any m . Moreover the commutator subgroup of $N_2(S)$ can be computed by applying Lemma 3.2 to the nil_2 -map η :

$$[(g_i^a g_{i+r}^b), (g_i^c g_{i+r}^d)] = [(g_j), (g_{j+r})]^{ad-bc}. \quad (2)$$

Therefore the commutator subgroup is a cyclic group generated by $[(g_i), (g_{i+r})]$, and it is contained in the center of the group. To understand the structure of $N_2(S)$ we prove some relations between the elements

$$k_i = (g_i g_{i+r})^{-1} (g_i) (g_{i+r}).$$

By equation 2 these elements are central in $N_2(S)$.

Lemma 3.4. *For $1 \leq i \neq j \leq r$ the relation $k_i = k_j$ holds in $N_2(S)$.*

Proof. Using the relations in $N_2(S)$ we have

$$\begin{aligned} k_i k_j^{-1} &= (g_i g_{i+r})^{-1} (g_i) (g_{i+r}) (g_{j+r})^{-1} (g_j)^{-1} (g_j g_{j+r}) \\ &= (g_i g_{i+r})^{-1} (g_i g_{j+r}^{-1}) (g_{i+r} g_j^{-1}) (g_j g_{j+r}) \\ &= (g_i g_{i+r})^{-1} (g_i g_{j+r}^{-1} g_{i+r} g_j^{-1}) (g_j g_{j+r}) \\ &= (g_{i+r}^{-1} g_i^{-1} g_i g_{j+r}^{-1} g_{i+r} g_j^{-1}) (g_j g_{j+r}) \\ &= (g_{j+r}^{-1} g_j^{-1}) (g_j g_{j+r}) = 1. \end{aligned}$$

□

Therefore we can simply omit the subscripts, and write $k = k_i$.

Lemma 3.5. *For $m \geq 1$ the relation*

$$k^m = (g_i^m g_{i+r})^{-1} (g_i^m) (g_{i+r})$$

holds in $N_2(S)$.

Proof. This is proved by induction on m . For $m = 1$ the relation holds by definition. The result will follow from the computation

$$\begin{aligned} (g_i^a g_{i+r})^{-1} (g_i) (g_i^{a-1} g_{i+r}) &= (g_{j+r}) (g_i^a g_{i+r})^{-1} (g_{j+r}^{-1} g_i) (g_i^{a-1} g_{i+r} g_j^{-1}) (g_j) \\ &= (g_{j+r}) (g_i^a g_{i+r})^{-1} (g_{j+r}^{-1} g_i^a g_{i+r} g_j^{-1}) (g_j) \\ &= (g_j g_{j+r})^{-1} (g_j) (g_{j+r}) = k, \end{aligned}$$

where we used $[(g_{j+r}), (g_j g_{j+r})^{-1} (g_j)] = 1$. Now assuming the statement holds for $m = a - 1$ we can write

$$\begin{aligned} (g_i^a g_{i+r}) &= k^{-1} (g_i) (g_i^{a-1} g_{i+r}) \\ &= k^{-1} (g_i) k^{1-a} (g_i^{a-1}) (g_{i+r}) \\ &= k^{-a} (g_i^a) (g_{i+r}). \end{aligned}$$

□

Lemma 3.6. *The following relation holds in $N_2(S)$:*

$$(x)(y)(xy)^{-1} = k^\alpha \text{ for all } x, y \in \langle g_i, g_{i+r} \rangle,$$

where α is such that $[x, y] = c^\alpha$.

Proof. For $j \neq i$ we have

$$\begin{aligned} (x)(y) &= (g_{j+r})(g_{j+r}^{-1}x)(yg_j^{-\alpha})(g_j)^\alpha \\ &= (g_{j+r})(g_{j+r}^{-1}xyg_j^{-\alpha})(g_j)^\alpha \\ &= (xy)(g_{j+r})(g_{j+r}^{-1}g_j^{-\alpha})(g_j)^\alpha \\ &= (xy)(g_{j+r}^{-1}g_j^{-\alpha})(g_j)^\alpha(g_{j+r}) \\ &= (xy)(g_j^\alpha g_{j+r})^{-1}(g_j)^\alpha(g_{j+r}) \\ &= (xy)k^\alpha. \end{aligned}$$

In the last step we used Lemma 3.5. □

Observe that any element x in S can be written as a product $c^k x_1 x_2 \cdots x_r$ for some k where x_i is of the form $g_i^{a_i} g_{i+r}^{b_{i+r}}$ for some a_i, b_{i+r} . Moreover, $[x_i, x_j] = 1$ for all i, j . Therefore $(x) = (c^k)(x_1)(x_2) \cdots (x_r)$ since $[(x_i), (x_j)] = 1$ in $N_2(S)$.

Lemma 3.7. *The group $N_2(S)$ sits in a central extension*

$$1 \rightarrow \langle k \rangle \rightarrow N_2(S) \xrightarrow{\epsilon} S \rightarrow 1.$$

Proof. The kernel of ϵ is the normal closure of the subgroup generated by $(xy)^{-1}(x)(y)$ where $x, y \in S$. We can write $x = c^k \prod_{i=1}^r x_i$ and $y = c^l \prod_{i=1}^r y_i$ for some k, l , and $x_i, y_i \in \langle g_i, g_{i+r} \rangle$. Then we compute

$$\begin{aligned} (xy)^{-1}(x)(y) &= (\prod x_i y_i)^{-1} (\prod x_i) (\prod y_i) (c)^{-k-l} (c)^k (c)^l \\ &= \prod (x_i y_i)^{-1} \prod (x_i) \prod (y_i) \\ &= \prod (x_i y_i)^{-1} (x_i)(y_i) \\ &= \prod k^{\alpha_i} = k^{\alpha_1 + \alpha_2 + \cdots + \alpha_r} \end{aligned}$$

where α_i is such that $[x_i, y_i] = c^{\alpha_i}$. We used Lemma 3.6 in the last step. □

Theorem 3.8. *Let $\{g_i\}_{i=1}^{2r}$ be a non-trivial symplectic sequence in G for some $r \geq 2$, and S denote the subgroup generated by $\{g_i\}$. Then the natural map $N_2(S) \rightarrow D_2(S)$ is an isomorphism. Moreover $N_2(S) \rightarrow N_2(G)$ is injective, and its image is the subgroup generated by $\{(g_i)\}$.*

Proof. We want to show that the natural map

$$\bar{\epsilon} : N_2(S) \rightarrow D_2(S), \quad (x) \mapsto (x, x^{-1}),$$

is an isomorphism. This map is surjective by Proposition 2.3. We will show that it is also injective. Since $\bar{\epsilon}$ factors ϵ it suffices to show that $\ker(\epsilon) = \langle k \rangle$ maps injectively under $\bar{\epsilon}$. We have $\bar{\epsilon}(k) = (1, c)$. Assume $\bar{\epsilon}(k^m) = 1$, that is $c^m = 1$, for some m . In particular, this implies that g_i^m lies in the center of S for all i . By Lemma 3.5 we have

$$\begin{aligned} k^m &= (g_i^m g_{i+r})^{-1} (g_i^m) (g_{i+r}) \\ &= (g_{i+r})^{-1} (g_i^m)^{-1} (g_i^m) (g_{i+r}) = 1 \end{aligned}$$

since $[(g_i^m), (g_{i+r})] = 1$. This proves the injectivity of $\bar{\epsilon}$.

The second statement follows from the diagram

$$\begin{array}{ccc} N_2(S) & \longrightarrow & N_2(G) \\ \downarrow \cong & & \downarrow \\ D_2(S) & \longrightarrow & D_2(G) \end{array}$$

since the map $D_2(S) \rightarrow D_2(G)$ is injective. □

4. EXAMPLES AND APPLICATIONS

In this section we list some of the consequences of the group theoretic results obtained in the previous sections. Our main result implies that there exists groups G for which $B(2, G)$ is not a $K(\pi, 1)$ space. On the geometric side, the space $B(2, G)$ is a classifying space for principal G -bundles of transitional nilpotency class less than 2. A principal G -bundle over a CW-complex X is said to have transitional nilpotency class less than 2 if there exists a trivialization of the bundle by an open cover such that on intersections the transition functions commute when they are simultaneously defined. Two such bundles p_0 and p_1 are called 2-transitionally isomorphic if there exists a principal G -bundle p over $X \times [0, 1]$ with transitional nilpotency class less than 2 whose restrictions $p|_{X \times 0}$ and $p|_{X \times 1}$ give p_0 and p_1 , respectively [2, Definition 5.3]. One implication of having a non-trivial higher homotopy group $\pi_n(B(2, G))$ for some $n \geq 2$ is that there exists principal G -bundles over the n -sphere \mathcal{S}^n which are not 2-transitionally isomorphic to the trivial principal G -bundle. Note that in the case of ordinary principal G -bundles every principal G -bundle over \mathcal{S}^n is isomorphic to the trivial principal G -bundle when $n \geq 2$ since $\pi_n(BG) = 0$ for all $n \geq 2$.

Homotopy type of $B(2, G)$. The colimit $N_q(G)$ is isomorphic to the fundamental group of the geometric realization of a simplicial set $B(q, G) \subset BG$ introduced in [1]. This simplicial set can be described as a colimit of the classifying spaces of nilpotent subgroups of class less than q [1, Theorem 4.3], also see [5] for further properties. Similar to the usual classifying space BG the association $G \mapsto B(q, G)$ is functorial on the category of groups, moreover it extends to a functor

$$B(q, -) : \mathbf{Grp}_q \rightarrow \mathbf{S}$$

from the extended version of the category of groups as introduced in §2 to the category of simplicial sets.

The inclusion map $B(q, G) \rightarrow BG$ defines a principal G -bundle $E(q, G) \rightarrow B(q, G)$. Looking at the long exact sequence of homotopy groups associated to the fibration in low degrees we obtain a short exact sequence of groups

$$1 \rightarrow \pi_1(E(q, G)) \rightarrow N_q(G) \rightarrow G \rightarrow 1, \tag{3}$$

where we identify the fundamental group of $\pi_1(B(q, G))$ with the colimit $N_q(G)$. The characterization in Proposition 2.2 of abelian groups can be translated into the following statement.

Proposition 4.1. *A group G is abelian if and only if $E(2, G)$ is contractible.*

Proof. If G is abelian then by the description of $B(2, G)$ as a colimit of the classifying spaces BA of abelian subgroups A , we have an isomorphism $B(2, G) \cong BG$. Therefore the space $E(2, G)$ is isomorphic to the contractible space EG . Conversely, if the space $E(2, G)$ is contractible then the kernel of $N_2(G) \rightarrow G$ is trivial by 3. By Proposition 2.2 G is abelian. \square

More generally, one can ask when $B(q, G)$ has the homotopy type of a $K(\pi, 1)$ space, a question raised in [1]. There is a natural map $B(q, G) \rightarrow BN_q(G)$ as defined in [1, Theorem 4.4].

Proposition 4.2. *Suppose that G is a finite group. If the natural map $B(q, G) \rightarrow BN_q(G)$ is a homotopy equivalence then the kernel of $\epsilon : N_q(G) \rightarrow G$ is torsion free.*

Proof. Suppose that the map $B(q, G) \rightarrow BN_q(G)$ is a homotopy equivalence. This means that $B(q, G)$ is a $K(N_q(G), 1)$ space. Since $E(q, G)$ is a finite cover of $B(q, G)$ this implies that $E(q, G)$ is a $K(\pi, 1)$ space. The space $E(q, G)$ is homotopy equivalent to the nerve of the poset of right cosets $\{gN \subset G \mid N \in \mathcal{N}_q(G)\}$ of nilpotent subgroups of class less than q , ordered by inclusion [5, §3.3]. In particular, when G is finite the nerve of this poset is finite dimensional hence $E(q, G)$ has finite cohomological dimension. Therefore $E(q, G)$ has the homotopy type of a finite dimensional complex. Note that if a finite dimensional complex is a $K(\pi, 1)$ space then its fundamental group is torsion free. This follows from the fact that the cohomological dimension of π is less than or equal to its geometric dimension, and π is torsion free if its cohomological dimension is finite [3, Chapter VIII]. Therefore $\pi_1(E(q, G))$, which is isomorphic to the kernel of the map $N_q(G) \rightarrow G$, is torsion free. \square

Next we describe our main result on the homotopy type of $B(2, G)$. This theorem gives an algebraic condition on G which implies that $B(2, G)$ is not a $K(\pi, 1)$ space. First examples of such groups are given in [5] using different methods.

Theorem 4.3. *Suppose that G is a finite group which has a non-trivial symplectic sequence $\{g_i\}_{i=1}^{2r}$ for some $r \geq 2$. Then $B(2, G)$ is not a $K(\pi, 1)$ space.*

Proof. Let $S \subset G$ denote the subgroup generated by $\{g_i\}$. By Theorem 3.8 there is a commutative diagram

$$\begin{array}{ccc} N_2(S) & \hookrightarrow & N_2(G) \\ \downarrow \epsilon_S & & \downarrow \epsilon_G \\ S & \hookrightarrow & G \end{array}$$

and ϵ_S factors as $N_2(S) \cong D_2(S) \xrightarrow{\pi_1} S$. The kernel of π_1 is generated by the element $(1, c)$ which is of finite order since G is finite. Hence there exists a torsion element in the kernel of ϵ_G , and Proposition 4.2 implies that $B(2, G)$ cannot be a $K(\pi, 1)$ space. \square

Examples. We describe how to obtain a non-trivial symplectic sequence $\{g_i\}_{i=1}^{2r}$, $r \geq 2$, for the following groups so that Theorem 4.3 applies.

- (1) Extraspecial p -groups are central in the ideas developed in this paper, such as the definition of a symplectic sequence. An extraspecial p -group is a central extension

$$Z(P) = \mathbb{Z}/p \triangleleft E_r \rightarrow (\mathbb{Z}/p)^{2r}$$

where $Z(P)$ is also the commutator subgroup. The quotient group has the structure of a symplectic vector space when regarded as a vector space over \mathbb{F}_p with the commutator map inducing a non-degenerate alternating bilinear form. Assume that $r \geq 2$. The lift $\{\tilde{e}_i\}_{i=1}^{2r}$ of a symplectic basis $\{e_i\}_{i=1}^{2r}$ in the quotient group $(\mathbb{Z}/p)^{2r}$ gives a non-trivial symplectic sequence in E_r . Moreover, E_r is generated by $\{\tilde{e}_i\}$ hence by Theorem 3.8 we have $N_2(E_r) \cong D_2(E_r)$. Therefore the kernel of $N_2(E_r) \rightarrow E_r$ is cyclic of order p .

- (2) The general linear group $\mathrm{GL}_n(\mathbb{F}_q)$, $n \geq 4$, over a finite field \mathbb{F}_q of characteristic p has a symplectic sequence given by the elementary matrices

$$\{g_1 = E_{12}, g_2 = E_{13}, g_3 = E_{2n}, g_4 = E_{3n}\}$$

where E_{ij} has 1's on the diagonal and in the (i, j) -slot. It follows easily from the commutation relations of elementary matrices that this sequence satisfies the required commutation relations of a non-trivial symplectic sequence.

- (3) We can embed $\mathrm{GL}_n(\mathbb{F}_p)$ inside the symmetric group Σ_{p^n}

$$\iota : \mathrm{GL}_n(\mathbb{F}_p) \rightarrow \Sigma_{p^n}$$

by regarding linear transformations of an n -dimensional vector space V over \mathbb{F}_p as the permutations of a set of cardinality p^n . When $n = 4$ the image of the symplectic sequence $\{g_i\}$ obtained from elementary matrices in $\mathrm{GL}_4(\mathbb{F}_p)$ gives a non-trivial symplectic sequence in Σ_{p^4} . Therefore for $k \geq 2^4$ the natural inclusion $\Sigma_{2^4} \rightarrow \Sigma_k$ gives a non-trivial symplectic sequence in Σ_k . One can conclude similarly for the alternating group A_k by using the isomorphism $\mathrm{GL}_4(\mathbb{F}_2) \cong A_8$, and the natural inclusion $A_8 \rightarrow A_k$ where $k \geq 8$.

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